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Examples of isochronous Hamiltonians: classical and quantal treatments

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Abstract
Recently, a general technique has been introduced to Ω-modify a *Hamiltonian* so that the Hamiltonian thereby produced is, in the classical context, isochronous. In this paper, we introduce and discuss simple examples of isochronous Hamiltonians manufactured in this manner. We also outline their quantal treatment, yielding equispaced spectra.

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1. Introduction

Recently, various techniques have been introduced to modify a Hamiltonian so that the Hamiltonian thereby produced is isochronous (for a review see [1]). The most recent technique [2] can be applied to a quite general, real Hamiltonian $H(\underline{p}, \underline{q})$, yielding the Ω-modified real Hamiltonian

$$\tilde{H}(\underline{p}, \underline{q}; \Omega) = \frac{C}{2}\left\{\left[\frac{H(\underline{p}, \underline{q})}{C}\right]^2 + \Omega^2[\Theta(\underline{p}, \underline{q})]^2\right\}, \tag{1}$$

where C is an arbitrary (nonvanishing) real constant inserted merely to adjust the dimensions (it has of course the dimensionality of a Hamiltonian), Ω is a positive constant ($\Omega > 0$) to which the basic isochrony period

$$T = \frac{2\pi}{\Omega} \tag{2}$$

is associated, and $\Theta(\underline{p}, \underline{q})$ is a collective coordinate whose Poisson bracket with the unmodified Hamiltonian is unity,

$$[H, \Theta] \equiv \sum_{n=1}^{N} \left[\frac{\partial H(\underline{p}, \underline{q})}{\partial p_n} \frac{\partial \Theta(\underline{p}, \underline{q})}{\partial q_n} - \frac{\partial \Theta(\underline{p}, \underline{q})}{\partial p_n} \frac{\partial H(\underline{p}, \underline{q})}{\partial q_n} \right] = 1. \tag{3}$$

Of course here and hereafter the N-vectors $\underline{p} \equiv (p_1, \ldots, p_N)$, respectively, $\underline{q} \equiv (q_1, \ldots, q_N)$ have as their components the canonical momenta p_n, respectively, the canonical coordinates q_n, and the index n ranges from 1 to the arbitrary positive integer N.

It is plain [2] that the time-evolution yielded by the Ω Hamiltonian $\tilde{H}(\underline{p}, \underline{q}; \Omega)$, on which we hereafter focus, entails that the unmodified Hamiltonian $H(t) \equiv H[\underline{p}(t), \underline{q}(t)]$ and the collective coordinate $\Theta(t) \equiv \Theta[\underline{p}(t), \underline{q}(t)]$ evolve according to the equations of motion

$$\dot{\Theta} = \frac{H}{C}, \qquad \dot{H} = -C\Omega^2\Theta, \tag{4}$$

hence they evolve periodically with period T,

$$H(t) = H(0)\cos(\Omega t) - C\Theta(0)\Omega\sin(\Omega t), \tag{5a}$$

$$\Theta(t) = \Theta(0)\cos(\Omega t) + \frac{H(0)}{C}\frac{\sin(\Omega t)}{\Omega}. \tag{5b}$$

Here and below $H(0) \equiv H[\underline{p}(0), \underline{q}(0)]$ and $\Theta(0) \equiv \Theta[\underline{p}(0), \underline{q}(0)]$.

Likewise, the canonical coordinates and momenta $q_n(t)$ and $p_n(t)$ evolve according to the equations of motion yielded by the Hamiltonian $\tilde{H}(\underline{p}, \underline{q}; \Omega)$, reading

$$\dot{q}_n = \frac{H}{C}\frac{\partial H}{\partial p_n} + C\Omega^2\Theta\frac{\partial \Theta}{\partial p_n} = \dot{\Theta}\frac{\partial H}{\partial p_n} + C\Omega^2\Theta\frac{\partial \Theta}{\partial p_n}, \tag{6a}$$

$$\dot{p}_n = -\frac{H}{C}\frac{\partial H}{\partial q_n} - C\Omega^2\Theta\frac{\partial \Theta}{\partial q_n} = -\dot{\Theta}\frac{\partial H}{\partial q_n} - C\Omega^2\Theta\frac{\partial \Theta}{\partial q_n}; \tag{6b}$$

and this entails, via (5) [1, 2], that, up to the possible occurrence of singularities (to be analyzed on a case-by-case basis), these canonical coordinates evolve isochronously with period T,

$$q_n(t+T) = q_n(t), \qquad p_n(t+T) = p_n(t), \tag{7}$$

at least for a class of initial data $q_n(0)$ and $p_n(0)$ belonging to an *open* (hence fully-dimensional) sector of phase space—which might coincide with the entire phase space, thereby qualifying the corresponding Ω system as an entirely isochronous system.

In this paper, we introduce and discuss simple examples of isochronous Ω-modified Hamiltonians manufactured in this manner, and we then outline their quantal treatment. And let us end this introduction by pointing out that the isochronous Hamiltonian models treated in this paper are different from the models discussed in our two previous papers [3, 4], which were obtained by a somewhat different technique to manufacture isochronous Hamiltonians; and they are as well different from the isoperiodic and isochronous Hamiltonians systems discussed and reviewed in the recent paper by Asorey *et al* [5], see also the references quoted there.

2. Two simple examples with $N = 1$

For $N = 1$ the time evolution of the (single) canonical coordinate q and of the corresponding canonical momentum p can be directly obtained (without the need to perform any integration) by solving for $q \equiv q(t)$ and $p \equiv p(t)$ the two equations (see (5))

$$H(p, q) = H(t) = H(0)\cos(\Omega t) - C\Theta(0)\Omega\sin(\Omega t), \tag{8a}$$

$$\Theta(p, q) = \Theta(t) = \Theta(0)\cos(\Omega t) + \frac{H(0)}{C}\frac{\sin(\Omega t)}{\Omega}. \tag{8b}$$

It is moreover plain that in this case the Hamiltonian equations of motion

$$\dot{q} = \frac{H}{C}\frac{\partial H}{\partial p} + C\Omega^2\Theta\frac{\partial\Theta}{\partial p}, \qquad \dot{p} = -\frac{H}{C}\frac{\partial H}{\partial q} - C\Omega^2\Theta\frac{\partial\Theta}{\partial q} \tag{9}$$

yield, via (4) and (6), the *Newtonian* equation of motion

$$\begin{aligned}\ddot{q} = &\left\{\frac{\partial}{\partial q}\left[\frac{H}{C}\frac{\partial H}{\partial p} + C\Omega^2\Theta\frac{\partial\Theta}{\partial p}\right]\right\}\left[\frac{H}{C}\frac{\partial H}{\partial p} + C\Omega^2\Theta\frac{\partial\Theta}{\partial p}\right]\\ &- \left\{\frac{\partial}{\partial p}\left[\frac{H}{C}\frac{\partial H}{\partial p} + C\Omega^2\Theta\frac{\partial\Theta}{\partial p}\right]\right\}\left[\frac{H}{C}\frac{\partial H}{\partial q} + C\Omega^2\Theta\frac{\partial\Theta}{\partial q}\right],\end{aligned} \tag{10}$$

where of course, in the right-hand side, the coordinate p should be eliminated by solving for it, in terms of q and $\dot{q}$, the first Hamiltonian equation, see (9). Hence this Newtonian equation of motion shall generally yield an isochronous time evolution, with basic period T, see (2)—up to possible singularities introduced by the solution for p and q of equations (8).

2.1. *Example 1*

In this section we consider the simple example, identified by the assignments

$$H(p, q) = \omega pq, \qquad \Theta(p, q) = \frac{1}{\omega}\log\left(\frac{q}{a}\right), \tag{11}$$

where ω and a are two arbitrary positive constants. The compatibility of these definitions with (3) is plain.

The corresponding Ω-modified Hamiltonian reads (see (1))

$$\tilde{H}(p, q; \Omega) = \frac{1}{2}\left[\frac{\omega^2 q^2 p^2}{C} + \frac{C\Omega^2}{\omega^2}\log^2\left(\frac{q}{a}\right)\right], \tag{12a}$$

and it yields the Hamiltonian equations of motion

$$\dot{q} = \frac{\omega^2 q^2}{C}p, \qquad \dot{p} = -\frac{\omega^2 p^2}{C}q - \frac{C\Omega^2}{\omega^2 q}\log\left(\frac{q}{a}\right), \tag{12b}$$

and the Newtonian equation of motion

$$\ddot{q} = \frac{\dot{q}^2}{q} - \Omega^2 q\log\left(\frac{q}{a}\right). \tag{12c}$$

The solutions of these equations of motion read (see (11))

$$q(t) = a\exp[\omega\Theta(t)], \qquad p(t) = \frac{H(t)\exp[-\omega\Theta(t)]}{a\omega}, \tag{13a}$$

with $H(t)$ and $\Theta(t)$ evolving according to (5), so that

$$q(t) = a\left[\frac{q(0)}{a}\right]^{\cos(\Omega t)}\exp\left[\frac{\dot{q}(0)}{q(0)}\frac{\sin(\Omega t)}{\Omega}\right], \tag{13b}$$

$$p(t) = p(0)\left\{\cos(\Omega t) + \log\left[\frac{q(0)}{a}\right]\frac{a\Omega}{\dot{q}(0)}\sin(\Omega t)\right\}\left[\frac{q(0)}{q(t)}\right]. \tag{13c}$$

It is therefore plain that, for any initial data with $q(0) > 0$, both canonical variables are periodic with period T, and for all time the canonical coordinate $q(t)$ is positive, $q(t) > 0$.

2.2. Example 2

In this section, we consider another simple example identified by the assignments

$$H(p,q) = \frac{p}{g'(q)} + f(q), \tag{14a}$$

$$\Theta(p,q) = g(q) + \alpha H(p,q) = g(q) + \frac{\alpha p}{g'(q)} + \alpha f(q), \tag{14b}$$

yielding via (1),

$$\begin{aligned}\tilde{H}(p,q;\Omega) &= \frac{1}{2C}\{H^2 + C^2\Omega^2[\alpha H(p,q) + g(q)]^2\}\\ &= \frac{1}{2C}\{p^2(1+\alpha^2C^2\Omega^2)[g'(q)]^{-2}\\ &\quad + 2p\{\alpha C^2\Omega^2 g(q) + f(q)(1+\alpha^2C^2\Omega^2)\}[g'(q)]^{-1}\\ &\quad + C^2\Omega^2[g(q)+\alpha f(q)]^2 + [f(q)]^2\}.\end{aligned} \tag{15}$$

Here and below α is an *a priori* arbitrary constant, $f(q)$ and $g(q)$ are two *a priori* arbitrary functions of the canonical coordinate q, and primes denote differentiations with respect to the argument of the functions they are appended to. The fact that the definitions (14) satisfy condition (3) is plain.

In this case, the time evolution of the canonical coordinate $q \equiv q(t)$ is obtained by solving for this dependent variable the equation

$$g(q) = \Theta(t) - \alpha H(t), \tag{16a}$$

and likewise the time evolution of the canonical momentum $p \equiv p(t)$ is then provided by the formula

$$p(t) = g'[q(t)]\{H(t) - f[q(t)]\}, \tag{16b}$$

where of course $H(t)$ and $\Theta(t)$ evolve according to (8).

A tedious computation yields the following neat Newtonian equation satisfied by the canonical coordinate $q(t)$:

$$\ddot{q} = -\frac{g''(q)\dot{q}^2 + \Omega^2 g(q)}{g'(q)}; \tag{17}$$

and it is easily seen that this indeed entails that the quantity

$$g(t) \equiv g[q(t)] \tag{18a}$$

evolves according to the ODE

$$\ddot{g} + \Omega^2 g = 0, \tag{18b}$$

consistently with (16*a*) and (8). Note that the time evolution of the canonical coordinate $q(t)$ turns out to be altogether independent of the function $f(q)$, although this function does affect the definition of the Hamiltonian $\tilde{H}(p,q)$, see (15), and the time evolution of the canonical momentum $p(t)$, see (16*b*).

For instance for

$$g(q) = Aq^k \tag{19a}$$

with k an odd positive integer and A an arbitrary (real) constant, the time evolution of the canonical coordinate $q(t)$ clearly remains bounded for all times and is periodic with period T

for arbitrary initial data (see (16*a*) and (8)), while the Hamiltonian $\tilde{H}(p, q; \Omega)$ reads

$$\begin{aligned}\tilde{H}(p, q; \Omega) = \frac{1}{2C}\{&(1+\alpha^2C^2\Omega^2)(Akq^{k-1})^{-2}p^2. \\ &+[\alpha C^2\Omega^2k^{-1}q + f(q)(1+\alpha^2C^2\Omega^2)(Akq^{k-1})^{-1}]2p \\ &+C^2\Omega^2[\alpha f(q)+Aq^k]^2+[f(q)]^2\},\end{aligned} \tag{19b}$$

and the corresponding Newtonian equation of motion reads as follows (see (17)):

$$\ddot{q} = -\frac{(k-1)\dot{q}^2}{q} - \frac{\Omega^2 q}{k}. \tag{19c}$$

Note however that, contrary to the Hamiltonian equations of motion, this runs into a divergence whenever $q(t)$ vanishes.

3. A simple unmodified Hamiltonian linear in the canonical momenta

In this section we report, without much commentary, the results corresponding to the following assignment of the unmodified Hamiltonian:

$$H(\underline{p}, \underline{q}) = \sum_{n=1}^{N} a_n p_n \left[\frac{\partial\Theta(\underline{q})}{\partial q_n}\right]^{-1}, \tag{20a}$$

where the N arbitrary (real) constants a_n are only restricted by the condition

$$\sum_{n=1}^{N} a_n = 1, \tag{20b}$$

and the collective coordinate $\Theta(\underline{q})$ is an *a priori arbitrary* function of the canonical coordinates q_n, but is independent of the canonical momenta p_n. Note that this definition guarantees automatically the validity of condition (3).

It is then a matter of trivial algebra to obtain the evolution equations (yielded of course by the Hamiltonian $\tilde{H}(\underline{p}, \underline{q}; \Omega)$, see (1) with (20)),

$$\dot{q}_n = \frac{a_n}{C}\left[\frac{\partial\Theta(\underline{q})}{\partial q_n}\right]^{-1} H = \frac{a_n}{C}\left[\frac{\partial\Theta(\underline{q})}{\partial q_n}\right]^{-1}\sum_{m=1}^{N} a_m p_m \left[\frac{\partial\Theta(\underline{q})}{\partial q_m}\right]^{-1}, \tag{21a}$$

$$\dot{p}_n = \frac{H}{C}\sum_{m=1}^{N}\left\{a_m p_m\left[\frac{\partial\Theta(\underline{q})}{\partial q_m}\right]^{-2}\left[\frac{\partial^2\Theta(\underline{q})}{\partial q_m\partial q_n}\right]\right\} - C\Omega^2\Theta(\underline{q})\frac{\partial\Theta(\underline{q})}{\partial q_n}, \tag{21b}$$

and the Newtonian equations of motion

$$\frac{\partial\Theta(\underline{q})}{\partial q_n}\ddot{q}_n + \sum_{m=1}^{N}\left[\dot{q}_n\dot{q}_m\frac{\partial^2\Theta(\underline{q})}{\partial q_n\partial q_m}\right] + \Omega^2 a_n\Theta(\underline{q}) = 0 \tag{22}$$

(obtained by time-differentiating the first version of (21) after multiplying it by $\partial\Theta(\underline{q})/\partial q_n$ and then by using the second of equations (4)).

For instance if

$$\Theta(\underline{q}) = \sum_{m=1}^{N} b_m q_m^{k_m}, \tag{23a}$$

where the N (real) constants b_m are arbitrary and the N (real) exponents k_m are as well arbitrary (but see below), then the Newtonian equations of motion (22) read

$$k_n b_n \left[q_n^{k_n-1} \ddot{q}_n + (k_n - 1) q_n^{k_n-2} \dot{q}_n^2 \right] + \Omega^2 a_n \sum_{m=1}^{N} b_m q_m^{k_m} = 0, \tag{23b}$$

and (consistently with these equations, see (23a) and the second of (4)) the first version of the Hamiltonian equations (21a) read

$$k_n q_n^{k_n-1} \dot{q}_n = \frac{a_n}{b_n} \frac{H(t)}{C}, \qquad n = 1, \ldots, N, \tag{23c}$$

yielding via (5)

$$q_n(t) = q_n(0) \left\{ 1 + \frac{a_n}{b_n} \frac{1}{[q_n(0)]^{k_n}} \left[\frac{H(0)}{C} \frac{\sin(\Omega t)}{\Omega} + \Theta(0)[\cos(\Omega t) - 1] \right] \right\}^{1/k_n}, \quad n = 1, \ldots, N. \tag{23d}$$

This solution is clearly (real and) periodic with period T: for arbitrary (real) initial data if all the exponents k_m are rational numbers, $k_m = \nu_m/\delta_m$, with the two coprime integers ν_m and δ_m both positive and odd; otherwise the solution is (real and) *periodic* with period T only for initial data (if any) such that the data $[q_m(0)]^{k_m}$ are all real and

$$\left| \frac{b_n}{a_n} [q_n(0)]^{k_n} - \Theta(0) \right|^2 > [\Theta(0)]^2 + \left[\frac{H(0)}{C\Omega} \right]^2, \qquad n = 1, \ldots, N, \tag{23e}$$

namely, for every value of $n = 1, \ldots, N$, one of the following two inequalities holds:

$$\frac{b_n}{a_n} [q_n(0)]^{k_n} \gtrless \Theta(0) \pm \left\{ [\Theta(0)]^2 + \left[\frac{H(0)}{C\Omega} \right]^2 \right\}^{1/2}, \tag{23f}$$

where the double inequality symbols and the double signs in the right-hand side go of course always together (but this assignment could be different for different values of n).

4. The quantum case

In this section, we outline the quantal treatment of the two isochronous models of section 2. The standard quantization procedure we employ is to replace the coordinate q with the variable x and the momentum p with the differential operator $-\mathrm{i}\hbar d/\mathrm{d}x$ in the Hamiltonian $\tilde{H}(p, q; \Omega)$, thereby transforming it into the differential operator

$$\tilde{H}(p, q; \Omega) = \tilde{H}_Q \left(-\mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}x}, x; \Omega \right). \tag{24}$$

Here of course $\hbar$ is Planck's constant, and a suitable choice must be made of the ordering of the two noncommuting quantities x and $\mathrm{d}/\mathrm{d}x$, and also of the functional space on which they operate, so as to guarantee that this operator be self-adjoint. We then try and solve the stationary Schrödinger equation

$$\tilde{H}_Q \left(-\mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}x}, x; \Omega \right) \psi(x) = E\psi(x), \tag{25}$$

to see whether the hunch that the corresponding spectrum is discrete and equispaced with spacing $\hbar\Omega$ is confirmed.

As we shall see, ordering issues play a major role in this problem. For example, assume that quantizing the functions H and Θ by operators H_Q and Θ_Q, respectively, has been done in such a way that the following condition is satisfied:

$$e^{i\alpha H_Q}\, e^{i\beta\Theta_Q} = e^{i\alpha\beta}\, e^{i\beta\Theta_Q}\, e^{i\alpha H_Q}. \tag{26}$$

These are the so-called Weyl commutation relations and they guarantee the existence of a unitary operator U such that

$$UH_QU^{-1} = \frac{1}{i}\frac{d}{dx}, \tag{27}$$

$$U\Theta_QU^{-1} = x. \tag{28}$$

Under these circumstances it is straightforward to show that the quantum Hamiltonian $\tilde{H}_Q$ defined by

$$\tilde{H}_Q = \frac{1}{2C}H_Q^2 + \frac{C\Omega^2}{2}\Theta_Q^2 \tag{29}$$

is unitarily equivalent to the harmonic oscillator, since $U\tilde{H}_QU^{-1}$ is the Hamiltonian of the harmonic oscillator. The spectra therefore coincide and the eigenfunctions of $\tilde{H}_Q$ are mapped by U on those of the harmonic oscillator. Thus, if for example 1 we were to choose

$$H_Q = \frac{\omega}{2}\left(\frac{1}{i}\frac{d}{dx}x + x\frac{1}{i}\frac{d}{dx}\right), \tag{30a}$$

$$\Theta_Q = \frac{1}{\omega}\log\left(\frac{x}{a}\right), \tag{30b}$$

and were to define the corresponding quantum Hamiltonian via (29), then the entire issue of determining the spectrum would be trivial: the spectrum would be the same as that of the harmonic oscillator and the eigenfunctions determined via the unitary transformation, which is not difficult to establish. Similarly, for example 2, we could choose

$$H_Q = \frac{1}{2}\left[\frac{1}{i}\frac{d}{dx}\frac{1}{g'(x)} + \frac{1}{g'(x)}\frac{1}{i}\frac{d}{dx}\right] + f(x), \tag{31a}$$

$$\Theta_Q = g(x) + \alpha H_Q. \tag{31b}$$

Clearly the Weyl commutation relations are again satisfied and the operator defined through (29) via (31) clearly has the spectrum of the harmonic oscillator.

It is therefore enough to find quantizations of H and Θ satisfying the Weyl commutation relations in order to be sure that one can indeed define a Hamiltonian that is equivalent to the harmonic oscillator. Since H and Θ are canonically conjugate at the classical level, and since the Weyl commutation relations correspond to a similar property in quantum mechanics, the requirement is simply to find an ordering prescription for H and Θ of such a nature that their conjugate nature translates into a similar commutation relation.

In the following, we shall consider orderings which cannot obviously be written in the form (29). We shall instead see that, for example 1, our ordering yields an equispaced spectrum with the spacing corresponding to the harmonic oscillator frequency $\hbar\Omega$, but a different zero-point energy, whereas for a similar ordering in example 2 with $g(q) = Aq^{k-1}$ (see (19)) the energy spacing turns out to be $2\hbar\Omega$, but each eigenfunction is doubly degenerate. The number of eigenvalues less than a given energy E coincides asymptotically with that of the harmonic oscillator as $E \to \infty$, which is natural since that quantity is entirely determined by the classical behavior through the Weyl formula [6].

4.1. Quantal treatment of example 1

In view of the simplicity of this example the treatment given below is quite terse.

The stationary Schrödinger equation (25) corresponding to this example reads (see (12*a*))

$$\frac{1}{2}\left[-\frac{\hbar^2\omega^2}{C}\frac{\mathrm{d}}{\mathrm{d}x}x^2\frac{\mathrm{d}}{\mathrm{d}x}+\frac{C\Omega^2}{\omega^2}\log^2\left(\frac{x}{a}\right)\right]\psi(x)=E\psi(x), \tag{32}$$

and, consistently with our classical treatment (see the last sentence of section 2.1), we shall consider it in the interval $0 \leqslant x < \infty$. Note the quite specific choice of ordering for the operator corresponding to x^2p^2. While it certainly is a rather 'natural' choice, it does not correspond to the Weyl quantization prescription [7] nor to the quantization suggested above.

It is then a matter of trivial algebra to verify that the normalizable solutions of this stationary Schrödinger equation read

$$\psi_n(x)=\exp\left(-\frac{\xi}{2\lambda}-\frac{\xi^2}{2}\right)H_n(\xi),\qquad \xi=\lambda\log\left(\frac{x}{a}\right),\qquad \lambda=\left(\frac{C\Omega}{\hbar\omega^2}\right)^{1/2},$$
$$n=0,1,\ldots, \tag{33}$$

where $H_n(\xi)$ is the Hermite polynomial of order n, and the corresponding energy spectrum reads

$$E_n=\hbar\Omega\left(n+\frac{5}{8}\right),\qquad n=0,1,\ldots. \tag{34}$$

4.2. Quantal treatment of example 2

We now consider the special case (19) of example 2. We limit ourselves to the case of positive, odd integer values of *k*, and we consider the case $k=1$ separately, because it can be trivially reduced to the harmonic oscillator by introducing the change of dependent variable

$$\chi(x)=\mathrm{e}^{-\mathrm{i}F(x)/\hbar}\psi(x) \tag{35a}$$

where $F'(x)=f(x)$.

On the other hand, for $k=3,5,\ldots$, it is readily seen from the classical solution that the velocity of the solution when it crosses the origin is always infinite. This suggests, from semiclassical considerations, that $\psi(x)$ always vanishes at the origin, $\psi(0)=0$, whereas, of course, for $k=1$ this only holds for the odd states. We shall hence as a start limit ourselves to calculating the wavefunction on the positive real axis and later discuss the issue of solutions on the negative real axis.

In this case, we start from the following quantum Hamiltonian:

$$\begin{aligned}\tilde{H}_Q=\frac{1}{2C}&\left[\frac{\hbar}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{g'(x)}+f(x)\right]\left[\frac{1}{g'(x)}\frac{\hbar}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}x}+f(x)\right]\\&+\frac{C\Omega^2}{2}\left[\frac{\hbar\alpha}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{g'(x)}+\alpha f(x)+g(x)\right]\left[\frac{1}{g'(x)}\frac{\hbar\alpha}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}x}+\alpha f(x)+g(x)\right].\end{aligned} \tag{36}$$

We shall display explicitly a series of unitary transformations which lead from (36) to a harmonic oscillator with an additional $[g(q)]^{-2}$ potential.

First, one notes that it is possible to eliminate the function $f(q)$ altogether via the transformation

$$\psi_1(x)=\mathrm{e}^{-\mathrm{i}F(x)/\hbar}\psi(x), \tag{37}$$

where $F(x)$ is defined by

$$F'(x)=\frac{f(x)}{g'(x)}. \tag{38}$$

The modified Hamiltonian then reads

$$\tilde{H}_Q^{(1)} = -\frac{\hbar^2}{2C}\frac{\mathrm{d}}{\mathrm{d}x}[g'(x)]^{-2}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{C\Omega^2}{2}\left\{\frac{\hbar\alpha}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}x}[g'(x)]^{-1} + g(x)\right\}\left\{[g'(x)]^{-1}\frac{\hbar\alpha}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}x} + g(x)\right\}. \tag{39}$$

We now perform the following change of both dependent and independent variables:

$$\xi = g(x), \tag{40}$$

$$h(\xi) = g'(x), \tag{41}$$

$$\psi_2(\xi) = \frac{\psi(x)}{\sqrt{g'(x)}}. \tag{42}$$

The resulting Hamiltonian is

$$\tilde{H}_Q^{(2)} = -\frac{\hbar^2}{2C}\sqrt{h(\xi)}\frac{\mathrm{d}}{\mathrm{d}\xi}\frac{1}{h(\xi)}\frac{\mathrm{d}}{\mathrm{d}\xi}\sqrt{h(\xi)} + \frac{C\Omega^2}{2}\left[\sqrt{h(\xi)}\frac{\hbar\alpha}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}\xi}\frac{1}{\sqrt{h(\xi)}} + \xi\right]\left[\frac{1}{\sqrt{h(\xi)}}\frac{\hbar\alpha}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}\xi}\sqrt{h(\xi)} + \xi\right]. \tag{43}$$

This can be rewritten in a simpler way as

$$\tilde{H}_Q^{(2)} = \frac{1}{2C}D_+D_- + \frac{C\Omega^2}{2}(\alpha D_+ + \xi)(\alpha D_- + \xi), \tag{44}$$

where $D_\pm$ are defined as follows:

$$D_+ = \sqrt{h(\xi)}\frac{\hbar}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}\xi}\frac{1}{\sqrt{h(\xi)}} = \frac{\hbar}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}\xi} - \hbar\frac{h'(\xi)}{2ih(\xi)}, \tag{45a}$$

$$D_- = \frac{1}{\sqrt{h(\xi)}}\frac{\hbar}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}\xi}\sqrt{h(\xi)} = \frac{\hbar}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}\xi} + \hbar\frac{h'(\xi)}{2ih(\xi)}. \tag{45b}$$

At this stage we cannot proceed further in the general case. Making the same assumption as in the classical case, see (19a), that $g(q) = Aq^k$, leads to the following expression for $h(\xi)$:

$$h(\xi) = kA^{1-\gamma}\xi^\gamma, \qquad \gamma = \frac{k-1}{k}, \tag{46}$$

from which the following expressions for $D_\pm$ immediately follow:

$$D_\pm = \frac{\hbar}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}\xi} \mp \frac{\hbar\gamma}{2\mathrm{i}\xi}. \tag{47}$$

We now make a final transformation, defining

$$\psi_3(\xi) = \mathrm{e}^{-\mathrm{i}\lambda\xi^2/(2\hbar)}\psi_2(\xi), \tag{48a}$$

where λ is defined as

$$\lambda = \frac{\alpha C^2\Omega^2}{1 + C^2\alpha^2\Omega^2}. \tag{48b}$$

This leads to the final Hamiltonian

$$\tilde{H}_Q^{(3)} = -\frac{\hbar^2}{2C_1}\frac{d^2}{\mathrm{d}\xi^2} + \frac{C_1\Omega^2\xi^2}{2} + \frac{\gamma}{2}\left(1 + \frac{\gamma}{2}\right)\frac{\hbar^2}{2C_1\xi^2}, \tag{49a}$$

where C_1 is defined as

$$C_1 = \frac{C}{1 + C^2\alpha^2\Omega^2}. \tag{49b}$$

Finally, we perform the following cosmetic change of variable:

$$\psi_3(\xi) = \chi(\eta), \qquad \eta = \left(\frac{2C_1\Omega}{\hbar}\right)^{1/2}\xi, \tag{50a}$$

which is easily seen to imply that the new dependent variable η is dimensionless, and yields the following version of the stationary Schrödinger equation:

$$-\chi''(\eta) + \left[\frac{1}{4}\eta^2 + \frac{\gamma}{2}\left(1 + \frac{\gamma}{2}\right)\frac{1}{\eta^2}\right]\chi(\eta) = \frac{E}{\hbar\Omega}\chi(\eta). \tag{50b}$$

And it is now easily seen that the normalizable solutions of this equation read

$$\chi_n(\eta) = \exp\left(-\frac{\eta^2}{4}\right)\eta^{\gamma/2+1}L_n^{(\frac{\gamma+1}{2})}\left(\frac{\eta^2}{2}\right), \qquad n = 0, 1, \ldots, \tag{51a}$$

where $L_n^{(\frac{\gamma+1}{2})}(z)$ is the generalized Laguerre polynomial [8], and the corresponding (equispaced) spectrum reads

$$E_n = 2n + \frac{1}{2k} = 2n + \frac{1-\gamma}{2}, \qquad n = 0, 1, \ldots. \tag{51b}$$

Note that the appearance of a η^{-2} term in the Hamiltonian for $\chi(\eta)$ is entirely due to ordering issues. In particular, it is easily checked that for the quantization given in (29) via (31), no such term arises. However, our semiclassical intuition that the wavefunction must vanish at the origin is correct in either case: due to (42) the wavefunction $\psi(x)$ always vanishes at the origin. However, for the present choice of ordering, it vanishes more strongly, due to the η^{-2} term. In particular, no crossing between positive and negative real axis is allowed in this case. The spectrum defined by (51b) is thus everywhere twice degenerate. The behavior for positive and negative values of x is therefore altogether independent. Compared to the classical situation, where the particle crosses the origin and needs a period $2\pi/\Omega$ to return to its original condition, in the quantum case the particle is reflected at the origin, with the consequence that the quantum system returns to its initial state in half the time necessary for the classical system to do so. This therefore gives a transparent physical picture of the phenomenon involved in the appearance of a factor of two in the quantum-mechanical spectrum with respect to the corresponding classical dynamics: the rebound at the origin caused by the η^{-2} term leads to a period of the quantum system which is half as large as the corresponding classical one. Note further that, since the spectrum of (50b) differs from the spectrum of the standard harmonic oscillator, the original Hamiltonian (36) cannot be written in the form (29) with operators H_Q and Θ_Q satisfying the Weyl commutation relations (26).

Note that, up to now, we have always assumed implicitly that k is a positive odd integer, $k = 3, 5, \ldots$. A few remarks concerning the case $k = 1$ are in order: clearly, in this case, everything in the derivation of (49a) goes through. However, since in this case $\gamma = 0$, the η^{-2} term vanishes and (51b) does not provide a full solution of the eigenvalue problem. In this case, as is easy to see, the eigenvalue spacing will be $\hbar\Omega$ as usual.

Finally, it is amusing to note how, in the Hamiltonian described above, an orbit which is classically allowed is reflected back in quantum mechanics: it is as though a 'classically accessible' region became quantum mechanically inaccessible, in remarkable contrast to, say, the usual tunnel effect.

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